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Thermodynamic optimization of cooling techniques for electronic packages

A. BEJAN and G. A. LEDEZMA

Department of Mechanical Engineering and Materials Science, Duke University, Durham, NC 27708-0300, U.S.A.

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Abstract—This paper describes several fundamental trade offs that govern the optimization of cooling techniques for heat generating electric devices. Five basic cooling configurations are optimized. It is shown that in forced air cooling above room temperature the fan power requirement is minimum when the heat transfer contact area is optimized to a value that is of the order of $10^2 A_f$, where A_f is the flow cross-sectional area. This optimal contact area is independent of whether the coolant is mixed or unmixed inside the heat generating enclosure. In air cooling with natural updraft, the heat generation rate (or amount of electronics, or overall thermal conductance) is maximum when the heat transfer contact area is of order $10^2 A_f$. In forced convection cooling at cryogenic temperatures, the refrigerator power requirement has minima with respect to the cold-end (refrigeration load) temperature and the heat transfer contact area. The optimal heat transfer area is again of the order of $10^2 A_f$.

1. INTRODUCTION

The development of cooling techniques for electronic components and packages is one of the most important contemporary applications of fundamental heat transfer. The vast literature devoted to this application has been reviewed on several occasions [1–5]. In brief, the 'cooling problem' consists of improving the thermal contact between the heat generating parts of the electric apparatus and the cooling agent. This can be accomplished by selecting the heat transfer mechanism (e.g. forced air vs a boiling liquid), the shape and orientation of the coolant flow (e.g. natural convection vs jet impingement), and the relative geometric arrangement of components in packages (e.g. the optimal spacing between printed circuit boards in a package [6–8]).

In this paper we propose to examine the electronics cooling problem from the point of view of the electric power demanded by the cooling technique itself. This question must be addressed at the most fundamental level, because even in air cooled electronics above room temperature the fan power increases with the size of the apparatus (the noise also increases with the fan power). By 'size' we mean the amount of electronics or total heat generation rate $\dot{Q}$ that is installed in a given enclosure.

The challenge to minimize the power required by the cooling technique is greater in applications below room temperature, where it is necessary not only to pump the coolant through the electronic components but also to precool the agent in a refrigeration machine. Applications of this kind are the computers contemplated for operation at low (cryogenic) temperatures [9, 10]. Important applications are also the superconducting devices operating at high and, especially, low temperatures [11–13]. The potential savings in electric power consumption are great in applications of very large scales, such as stadium size superconducting coils for magnetic energy storage and plasma containment vessels for fusion reactors.

The present study was structured as a sequence of five fundamental problems built on simple models that capture the most basic characteristics of the electric systems and their cooling techniques. The models proceed from the simple toward the complex. The objective is to uncover the most fundamental optimization principles (or design tradeoffs) that can be put to practical use in real applications. The method of analysis and optimization is the combination of heat transfer, thermodynamics and fluid mechanics described in refs. [14, 15], and used subsequently in many engineering applications, for example, heat transfer augmentation [16, 17] and convective combustion [18].

2. FORCED AIR COOLING: THE WELL MIXED COOLANT MODEL

Consider first the fan power required for cooling with room temperature air (T_0, P_0) an enclosure filled with heat generating electronics. The maximum operating temperature of the electronic components (T_e) and their total power dissipation ($\dot{Q}$) are specified. The fan operates irreversibly. In the arrangement shown in Fig. 1 the fan is positioned downstream of the electronics because this is the more practical and more common design employed in current air cooling technology. The optimization principle presented in this section, however, applies unchanged to designs

NOMENCLATURE

A	heat transfer contact area
A_f	flow cross-sectional area
b	dimensionless group, equation (31)
Bo_H	Boussinesq number
c_p	specific heat at constant pressure
f	friction factor
g	gravitational acceleration
G	function, equation (39)
H	height (Fig. 3)
H	function, equation (39)
k	thermal conductivity
L	flow path length
$\dot{m}$	mass flow rate
M	dimensionless flow rate, equation (30)
$\hat{M}$	dimensionless flow rate, equation (35)
n	exponent, equation (18)
p	perimeter of thermal contact
P_0	ambient pressure
$\dot{Q}$	heat generation rate
$\hat{Q}$	dimensionless heat generation rate, equation (36)
R	ideal gas constant
$\dot{S}_{gen}$	entropy generation rate
St	Stanton number
T_e	operating temperature
T_0	ambient temperature
U	heat transfer coefficient
V	mean velocity
$\dot{W}$	power
$\tilde{W}$	dimensionless power, equation (30)
x	flow path length, equation (14)
x	area ratio, equation (38).

Greek symbols

β	coefficient of volumetric thermal expansion
ΔP	pressure drop
η_{II}	second law efficiency
ρ	density
τ_e, τ_{in}	dimensionless temperatures, equation (29).

Subscripts

$(\ )_{in}$	inlet
$(\ )_m$	minimized once
$(\ )_{max}$	maximum
$(\ )_{min}$	minimum
$(\ )_{mix}$	well mixed
$(\ )_{mm}$	minimized twice
$(\ )_{opt}$	optimal
$(\ )_{out}$	outlet
$(\ )_{rev}$	reversible limit
$(\ )_0$	reference case.

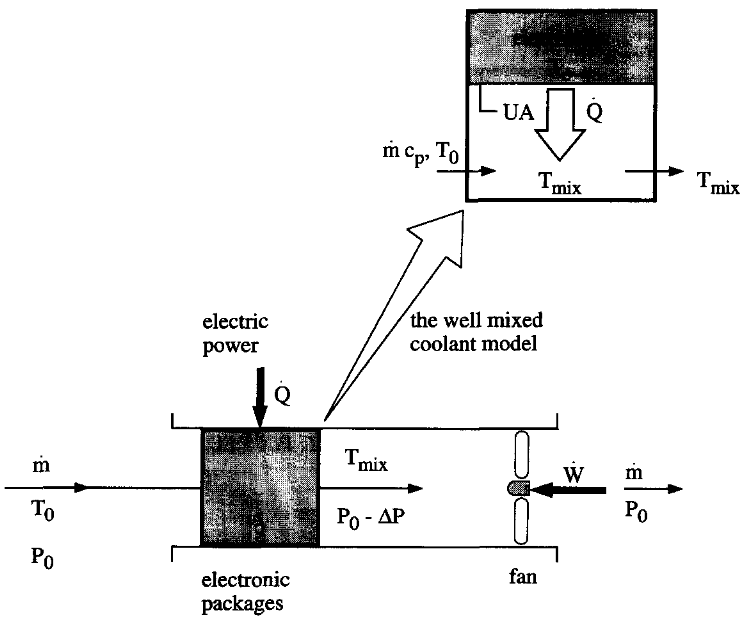

Fig. 1. The well mixed coolant model for an enclosure with heat generation and forced air cooling.

where the fan is positioned upstream of the electronic packages.

For simplicity, we begin with the assumption that the enclosure design and the air flow path are such that the air is well mixed at one bulk temperature (T_{mix}) inside the enclosure. We also assume that the surfaces of the electronic components are at the same temperature, T_e. Accordingly, the cooling rate provided by the air stream is

$$\dot{Q} = UA(T_e - T_{\rm mix}) \tag{1}$$

where A is the total heat transfer area of the components, and U is the average heat transfer coefficient based on A. The cooling rate is also equal to the enthalpy gained by the air stream,

$$\dot{Q} = \dot{m}c_{\rm p}(T_{\rm mix} - T_0). \tag{2}$$

Eliminating $T_{\rm mix}$ between equations (1) and (2) we obtain an important relation between the needed air flow rate and the cooling performance demanded by T_e and $\dot{Q}$

$$\frac{1}{\dot{m}c_{\rm p}} + \frac{1}{UA} = \frac{T_e - T_0}{\dot{Q}}. \tag{3}$$

The required fan power $\dot{W}$ is proportional to the product $\dot{m}\Delta P$, where ΔP is the air pressure drop across the package. We can show this by writing

$$\dot{W} = \frac{1}{\eta_{\rm II}}\dot{W}_{\rm rev} = \frac{1}{\eta_{\rm II}}\dot{m}c_{\rm p}(T_{\rm out,rev} - T_{\rm mix}) = \frac{1}{\eta_{\rm II}}\dot{m}c_{\rm p}T_{\rm mix}\left(\frac{T_{\rm out,rev}}{T_{\rm mix}} - 1\right) \tag{4}$$

where $\eta_{\rm II}(<1)$ is the known second law efficiency of the fan. The isentropic temperature ratio across the fan is

$$\frac{T_{\rm out,rev}}{T_{\rm mix}} = \left(\frac{P_0}{P_0 - \Delta P}\right)^{R/c_{\rm p}}. \tag{5}$$

Finally, we recognize that $\Delta P \ll P_0$, and that when expressed in Kelvin $T_{\rm min}$ is comparable with T_0. In this way equation (4) may be linearized to show that the actual fan power is indeed proportional to $\dot{m}\Delta P$

$$\dot{W} = \left(\frac{RT_0}{\eta_{\rm II}P_0}\right)\dot{m}\Delta P. \tag{6}$$

When the pressure drop is dominated by the core formed by the flow through the many air passages between components, ΔP is related to the contact area A and the average flow cross-section $A_{\rm f}$ by [19]

$$\Delta P = f\frac{A}{A_{\rm f}}\frac{1}{2}\rho V^2 \tag{7}$$

where V is the mean air velocity

$$V = \frac{\dot{m}}{\rho A_{\rm f}}. \tag{8}$$

Similarly, we may express the average heat transfer coefficient $U = St\rho c_{\rm p}V$ as

$$UA = St\,\dot{m}c_{\rm p}\frac{A}{A_{\rm f}} \tag{9}$$

such that by eliminating UA between equations (3) and (9) we arrive at

$$\dot{m}c_{\rm p} = \frac{\dot{Q}}{T_e - T_0}\left(1 + \frac{1}{St}\frac{A_{\rm f}}{A}\right). \tag{10}$$

In conclusion, equations (7) and (10) show that the fan power requirement (or $\dot{m}\Delta P$) is proportional to the group

$$\dot{W} \sim f\frac{A}{A_{\rm f}^3}\left(1 + \frac{1}{St}\frac{A_{\rm f}}{A}\right)^3. \tag{11}$$

The friction factor (f) and Stanton number (St) depend in general on the flow rate. This dependence, however, is weak when the flow is turbulent and the flow passage is extremely rough, as in a space filled with protruding electronic components. For this reason, and in order to illustrate in closed form the geometric optimization principle that is recommended by equation (11), we first regard f and St as two dimensionless constants (e.g. surfaces FT-2 and FTD-2 in ref. [20], pp. 190–191). Clearly, the $\dot{W}$ function (11) decreases monotonically as the flow cross-section $A_{\rm f}$ increases. The cross-sectional area, however, is dictated by the overall size of the enclosure and can be regarded as fixed (for example, $A_{\rm f}$ is comparable with, but smaller than, the frontal area of the enclosure).

Important from a thermal design stand point is that the $\dot{W}$ function (11) has a minimum with respect to the total heat transfer area. Solving $\partial\dot{W}/\partial A = 0$ we conclude that the optimal heat transfer area for minimum fan power is

$$A_{\rm opt} = \frac{2}{St}A_{\rm f}. \tag{12}$$

The corresponding optimal flow rate obtained by combining equations (12) and (10) is

$$\dot{m}_{\rm opt} = \frac{3}{2}\frac{\dot{Q}}{c_{\rm p}(T_e - T_0)}. \tag{13}$$

In flow passages with large protrusions and turbulent flow St is consistently a number of order 10^{-2} (e.g. ref. [20]). According to equation (12) then the optimal cooling area for minimum fan power is of the order of $10^2\,A_{\rm f}$. This estimate is approximate, i.e. valid only in an order of magnitude sense. The important contribution of equations (11) and (12) is to show that the fan power requirement *can be* minimized with respect to the heat transfer area built into the thermal design of the electronic packages. More accurate estimates of $A_{\rm opt}$ can be developed by minimizing $\dot{W}$ of equation (11) while accounting (through empirical relations) for the actual dependence of f and St on the air flow rate and then using equation (10). We illustrate this procedure in the next section.

3. FORCED AIR COOLING: THE UNMIXED COOLANT MODEL

When the air flow is guided in a certain way through the enclosure (e.g. by using partitions), the stream warms up gradually and reaches its highest tem-

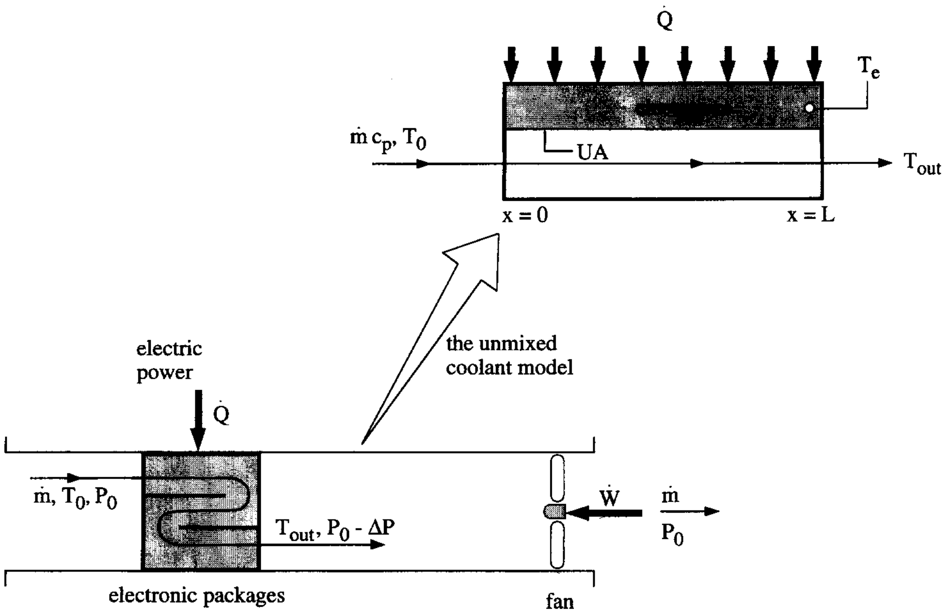

Fig. 2. The unmixed coolant model for an enclosure with heat generation and forced air cooling.

perature near the exit. If the electronic components generate heat nearly uniformly through the enclosure volume, the highest operating temperature occurs near the exit as well. This arrangement is illustrated in Fig. 2, where it is shown that the stream of path length L can be modeled as unmixed at the bulk temperature $T(x)$. Right at the exit we have

$$\frac{\mathrm{d}\dot{Q}}{\mathrm{d}x} = ph(T_e - T_{\mathrm{out}}) \tag{14}$$

where p is the perimenter of local contact, h is the heat transfer coefficient in the outlet cross-section, $T_{\mathrm{out}} = T(L)$, and the components temperature is at the highest operating level, T_e.

Intergrating equation (14) from $x = 0$ to $x = L$, we obtain

$$\dot{Q} = UA(T_e - T_{\mathrm{out}}) \tag{15}$$

where

$$UA = \int_0^L ph\,\mathrm{d}x. \tag{16}$$

The energy balance for the enclosure reads this time

$$\dot{Q} = \dot{m}c_{\mathrm{p}}(T_{\mathrm{out}} - T_0). \tag{17}$$

The key result is that by eliminating T_{out} between equations (15) and (17) we arrive again at equation (3) of the preceding section. The rest of that analysis continues to apply, and so does the conclusion that when f and St are constant the optimum is represented by equations (12) and (13). The optimization principle discussed so far appears to be independent of whether the coolant is mixed or unmixed inside the enclosure.

Consider now a more general case in which f and St vary with the flow rate. A reasonable turbulent flow model for such variations is

$$f = f_0\left(\frac{\dot{m}}{\dot{m}_0}\right)^{-n} \quad St = St_0\left(\frac{\dot{m}}{\dot{m}_0}\right)^{-n} \tag{18}$$

where the subscript 0 indicates a reference design. The exponent n is considerably smaller than 1, for example $n \cong 0.15$ in the case of the 'plain plate-fin surface 2.0' of ref. [20]. To review, the problem consists of minimizing the $\dot{W}$ function (11) with respect to A, while taking into account equations (3) and (18). It can be shown analytically that the optimal heat transfer area for minimum fan power is

$$A_{\mathrm{opt}} = \frac{2}{St_0} A_{\mathrm{f}} \left(\frac{\dot{m}_{\mathrm{opt}}}{\dot{m}_0}\right)^n \tag{19}$$

where $\dot{m}_{\mathrm{opt}}$ is given by exactly the same formula as in equation (13). It is important to note that the A_{opt} calculated with equation (19) is of the same order of magnitude as the estimate provided by equation (12), because $(\dot{m}_{\mathrm{opt}}/\dot{m}_0)^n$ is a factor of order 1 because n is much smaller than 1. The agreement between equations (12) and (19) is why, for the sake of clarity, in the next sections we continue to use the constant f and St assumptions.

4. AIR COOLING BY NATURAL DRAFT

The optimization principle presented until now for forced convection has an interesting counterpart in applications where the cooling mechanism is natural convection. In the latter, of course, there is no fan because the flow is driven by buoyancy. Instead, we will show that the total heat generation rate (or amount of electronics) can be maximized by properly selecting the total heat transfer area between components and air.

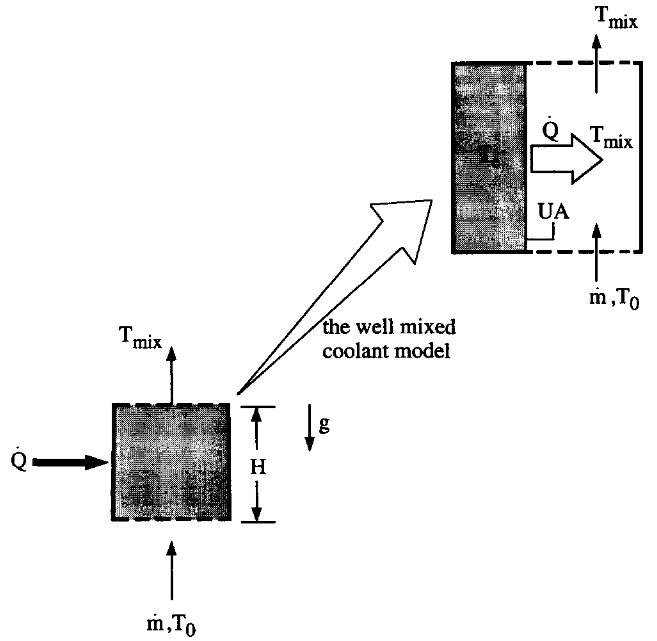

Fig. 3. Heat generating enclosure cooled by natural updraft.

Figure 3 shows an enclosure of height H, which contains components that generate heat at the total rate $\dot{Q}$. Room air (T_0) enters through the bottom of the enclosure, and exits through the top. The figure shows how the cooling effect can be modeled according to the well mixed coolant assumption. In this case the analysis begins with equations (1)–(3). The end result of the heat transfer analysis, however, is equation (10) regardless of whether the air stream is well mixed or unmixed.

The new feature in this technique is the effective pressure difference ΔP that drives the updraft. As shown in ref. [19] (Problem 6.7, p. 322), ΔP is the difference between the hydrostatic pressures at the bottom of two H-tall columns, one outside and filled with T_0 air, and the other inside and filled with T_{mix} air:

$$\Delta P = \rho g H \beta (T_{mix} - T_0). \tag{20}$$

Next, $(T_{mix} - T_0)$ can be eliminated between equations (20) and (2), and the ΔP result can be substituted into equation (7). Rearranging, we obtain an expression that varies as the inverse of the total heat generation rate:

$$\left(\frac{\rho c_p}{\dot{Q}}\right)^{2/3} (T_e - T_0)(2gH\beta)^{1/3} = f^{1/3}\frac{A^{1/3}}{A_f}\left(1 + \frac{1}{St}\frac{A_f}{A}\right). \tag{21}$$

The behavior of the function shown on the right hand side is most evident when we regard f and St as two constants. First, the total heat generation rate increases monotonically with the flow cross-sectional area A_f. Second, $\dot{Q}$ has a distinct maximum at an optimal contact area A_{opt} that is given by the same expression as in equation (12). This is a striking conclusion. It means that the optimization rule that in forced convection leads to minimum fan power requirement, in natural convection guarantees maximum installed electronics ($\dot{Q}$), or maximum overall thermal conductance. The maximum heat generation or cooling rate that corresponds to A_{opt} of equation (12) is

$$\frac{\dot{Q}_{max}}{kH(T_e - T_0)} = \left(\frac{2}{3}\right)^{3/2} Bo_H^{1/2}\left(\frac{St}{f}\right)^{1/2}\frac{A_f}{H^2} \tag{22}$$

where Bo_H is the Boussinesq number, $Bo_H = g\beta H^3 (T_e - T_0)/\alpha^2$. Equation (22) reconfirms the earlier finding that the cooling rate increases monotonically with the flow cross-sectional area A_f.

5. FORCED CONVECTION COOLING AT LOW TEMPERATURES

In this section we turn our attention to electric systems that must be maintained at temperatures below the ambient (Fig. 4). The cooling is by forced convection, and is described by equation (3) when the coolant is either mixed or unmixed inside the electric apparatus. The electric performance is specified by the Joule heating rate $\dot{Q}$ and maximum operating temperature T_e, which are fixed. In the following analysis we seek to determine the optimal coolant flow rate such that the refrigerator power requirement $\dot{W}$ is minimized.

For better illustration, we begin with the simplifying assumption that the irreversibility caused by the pressure drop across the electric apparatus is negligible among the other contributions to the power require-

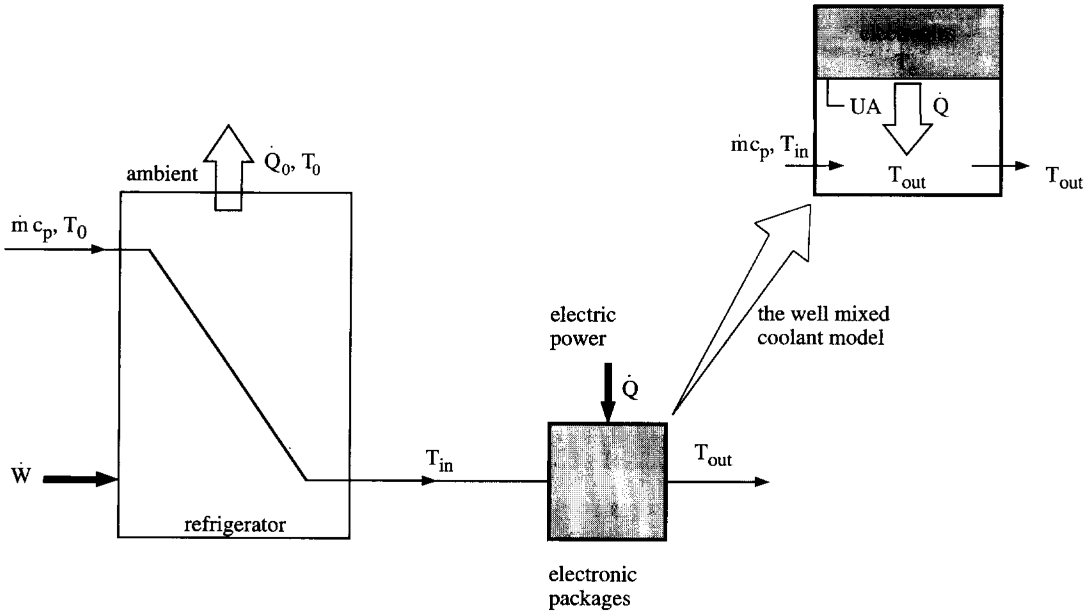

Fig. 4. Electric apparatus cooled by forced convection using a stream cooled in a refrigerator.

ment $\dot{W}$ (this assumption will be relaxed in the next section). As shown in Fig. 4, the inlet temperature of the coolant (T_{in}) is controlled by a steady-state refrigerator that receives room-temperature coolant (T_0), and rejects heat to the ambient ($\dot{Q}_0$, T_0). The coolant is an ideal gas such as cold nitrogen gas, or helium gas. The refrigerator operates irreversibly, $\dot{W} = \dot{W}_{rev}/\eta_{II}$, and its second law efficiency η_{II} is known. The power requirement in the reversible limit is obtained by eliminating $\dot{Q}_0$ between the first law and the second law for the refrigerator alone,

$$\dot{W}_{rev} = \dot{m}c_p(T_{in} - T_0) + \dot{Q}_0 \tag{23}$$

$$\dot{S}_{gen} = \dot{m}c_p \ln\frac{T_{in}}{T_0} + \frac{\dot{Q}_0}{T_0} = 0 \tag{24}$$

such that the result for the actual power requirement is

$$\dot{W} = \frac{1}{\eta_{II}}\dot{m}c_p\left(T_{in} - T_0 - T_0 \ln\frac{T_{in}}{T_0}\right). \tag{25}$$

A relation between $\dot{m}$ and T_{in} is provided by the heat transfer model [equation (3)], which reads

$$\frac{1}{\dot{m}c_p} + \frac{1}{UA} = \frac{T_e - T_{in}}{\dot{Q}}. \tag{26}$$

Equations (25) and (26) establish $\dot{W}$ as a function of $\dot{m}$. It is convenient to nondimensionalize these equations as

$$\tau_{in} = \tau_e - \frac{1}{M} \tag{27}$$

$$\tilde{W} = M(\tau_{in} - 1 - \ln\tau_{in}) \tag{28}$$

where

$$\tau_e = \frac{T_e}{T_0} \quad \tau_{in} = \frac{T_{in}}{T_0} \tag{29}$$

$$M = \frac{\dot{m}c_pT_0}{b\dot{Q}} \quad \tilde{W} = \frac{\eta_{II}\dot{W}}{b\dot{Q}} \tag{30}$$

and $U = St\rho c_p V$ and $V = \dot{m}/(\rho A_f)$. We also used the symbol b for the group

$$b = 1 + \frac{1}{St}\frac{A_f}{A}. \tag{31}$$

We assume that the flow passage geometry is given (A_f, A), and that St is relatively insensitive to changes in the flow rate.

The minimization of $\tilde{W}$ with respect to M begins with solving analytically $\partial\tilde{W}/\partial M = 0$, which yields an implicit relation for the optimal coolant inlet temperature as a function of the operating temperature of the electric system,

$$\frac{\ln\tau_{in,opt}}{1 - \tau_{in,opt}^{-1}} = \tau_e. \tag{32}$$

Combining this equation with equation (27) we obtain the optimal flow rate shown in Fig. 5. The existence of a $\dot{W}$ minimum with respect to flow rate can be explained as follows. When the flow rate is too small, the refrigeration temperature (T_{in}) must be low enough so that the coolant can remove the prescribed heat generation rate $\dot{Q}$ from the fixed level T_e. In this limit, $\dot{W}$ increases as T_{in} decreases because the Carnot-limit coefficient of performance decreases. In the opposite limit, when the flow rate is too large, the refrigerator power requirement increases because it is proportional to the flow rate. Because of these com-

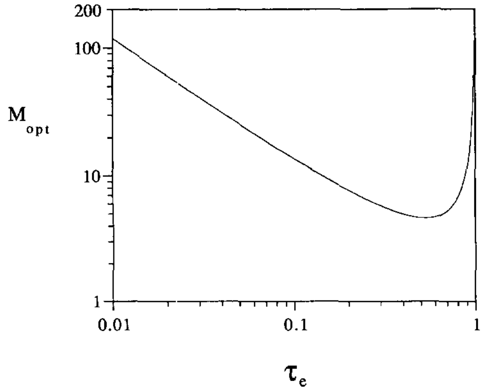

Fig. 5. The optimal coolant flow rate for the arrangement shown in Fig. 4, when the pressure drop irreversibility is negligible.

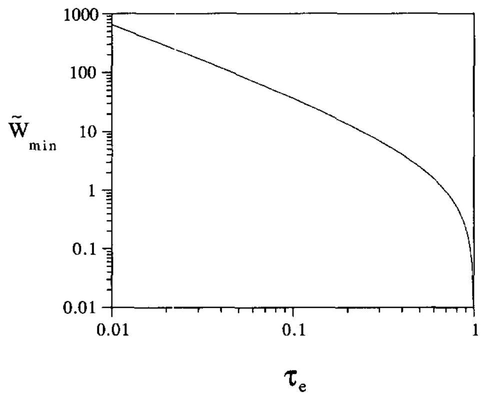

Fig. 6. The minimum refrigerator power requirement for the arrangement shown in Fig. 4, when the pressure drop irreversibility is negligible.

peting effects there is an intermediate flow rate where $\dot{W}$ is minimum.

Figure 5 also shows that the optimal flow rate has a minimum at $\tau_e = 0.53$, which corresponds to $T_e = 159$ K when $T_0 = 300$ K. Finally, by substituting M_{opt} of Fig. 5 and $\tau_{in,opt}$ of equation (32) into equation (28) we obtain the minimum refrigerator power plotted in Fig. 6. We see that $\tilde{W}_{min}$ first increases sharply as the operating temperature drops below the ambient temperature, and then increases more gradually as T_e approaches cryogenic temperatures.

The corresponding optimal coolant temperature furnished by equation (32) is shown in Fig. 7. The fact that $T_{in,opt}$ approaches T_0 when the operating temperature approaches T_0 explains why M_{opt} increases in the limit $\tau_e \to 1$ in Fig. 5: in this limit the effective temperature difference between components and coolant tends to zero, and, since $\dot{Q}$ is fixed, the only option is for the heat transfer coefficient (i.e. the flow rate) to increase.

6. COOLING AT LOW TEMPERATURES: THE OPTIMAL HEAT TRANSFER CONTACT AREA

In this section we examine the application of the heat transfer area optimization principle of Sections

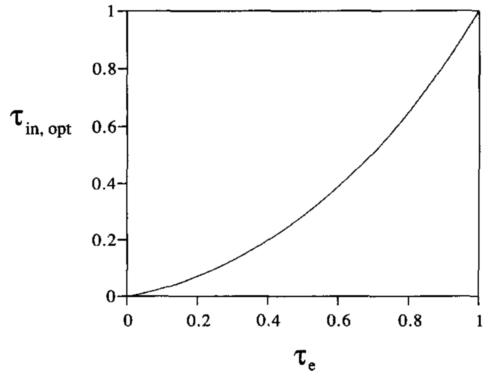

Fig. 7. The optimal coolant inlet temperature for the arrangement shown in Fig. 4 and optimized in Figs. 5 and 6.

2–4 to the forced convection cooling of a low temperature electric system. For this, it is necessary to take into account the irreversibility associated with the pressure drop ΔP experienced by the coolant as it flows through the heat generating electric apparatus. The cooling arrangement is the same as the one shown in Fig. 4 with the following pressures indicated along the coolant stream: P_0 at the inlet to the refrigerator, $P_0 + \Delta P$ at the refrigerator outlet (or inlet to the electric apparatus), and P_0 downstream of the electric apparatus.

The analysis follows the same steps as at the start of Section 5, except that equation (24) is replaced by a second law statement that accounts for the pressure rise ΔP experienced by the coolant inside the refrigerator,

$$\dot{S}_{gen} = \dot{m}c_p \ln\frac{T_{in}}{T_0} - \dot{m}R\ln\frac{P_0+\Delta P}{P_0} + \frac{\dot{Q}_0}{T_0} = 0. \tag{33}$$

Equations (23), (26) and (7) continue to hold. The coolant density in equation (7) is evaluated at the average temperature $(T_e + T_{in})/2$, while f and St continue to be modeled as sufficiently insensitive to changes in the flow rate. The refrigerator power requirement can be expressed as follows:

$$\frac{\eta_{II}\dot{W}}{\dot{Q}} = \hat{M}(\tau_{in} - 1 - \ln\tau_{in}) + \hat{M}^3\hat{Q}^2\frac{A}{A_f}(\tau_{in} + \tau_e) \tag{34}$$

where

$$\hat{M} = \frac{\dot{m}c_pT_0}{\dot{Q}} = \left(1 + \frac{1}{St}\frac{A_f}{A}\right)\bigg/(\tau_e - \tau_{in}) \tag{35}$$

and $\hat{Q}$ is a dimensionless group proportional to the specified heat generation rate $\dot{Q}$,

$$\hat{Q} = \frac{\dot{Q}f^{1/2}R}{2P_0A_fc_p^{3/2}T_0^{1/2}}. \tag{36}$$

For the purpose of optimizing the ratio $\eta_{II}\dot{W}/\dot{Q}$ numerically, it is convenient to combine equations (34) and (35) into

$$\frac{\eta_{\rm II}\dot{W}}{\dot{Q}} = \left(1+\frac{1}{xSt}\right)H + \left(1+\frac{1}{xSt}\right)^3 xG \qquad (37)$$

where x is the dimensionless heat transfer area,

$$x = \frac{A}{A_{\rm f}} \qquad (38)$$

and

$$H(\tau_{\rm in},\tau_e) = \frac{\tau_{\rm in}-1-\ln\tau_{\rm in}}{\tau_e-\tau_{\rm in}}$$

$$G(\tau_{\rm in},\tau_e,\hat{Q}) = \frac{\hat{Q}^2(\tau_{\rm in}+\tau_e)}{(\tau_e-\tau_{\rm in})^3}. \qquad (39)$$

The power requirement can be minimized in two ways, first, with respect to the heat transfer area (this tradeoff is the new aspect of this section), and, second, with respect to the refrigeration temperature $\tau_{\rm in}$ (this tradeoff was also encountered in Section 5). The minimization with respect to area is accomplished by solving $\partial(\eta_{\rm II}\dot{W}/\dot{Q})/\partial x = 0$ in conjunction with equation (37), which yields the following implicit formula for $x_{\rm opt}$ or $A_{\rm opt}/A_{\rm f}$:

$$(x_{\rm opt}St+1)^2\left(1-\frac{2}{x_{\rm opt}St}\right) = \frac{St\,H}{G}. \qquad (40)$$

Equation (40) shows that $x_{\rm opt}$ can be calculated when St, τ_e, $\tau_{\rm in}$ and $\hat{Q}$ are specified. By substituting numerically the $x_{\rm opt}$ function in place of x in equation (37) we obtain the once-minimized power requirement

$$\frac{\eta_{\rm II}\dot{W}_{\rm m}}{\dot{Q}} = \text{function}\,(St,\tau_e,\tau_{\rm in},\hat{Q}). \qquad (41)$$

The second minimization of the power requirement is executed numerically with respect to the refrigeration temperature $\tau_{\rm in}$ for an assumed set of values (St, τ_e, $\hat{Q}$). The end result is the optimal refrigeration temperature (Fig. 8) and the corresponding (twice-minimized) power requirement

$$\frac{\eta_{\rm II}\dot{W}_{\rm mm}}{\dot{Q}} = \text{function}\,(St,\tau_e,\hat{Q}). \qquad (42)$$

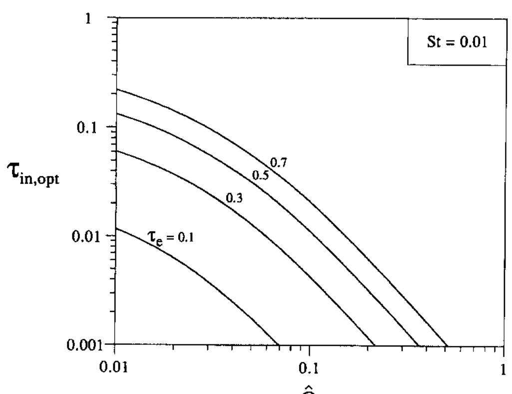

Fig. 8. The optimal coolant inlet temperature for forced convection with pressure drop (Section 6).

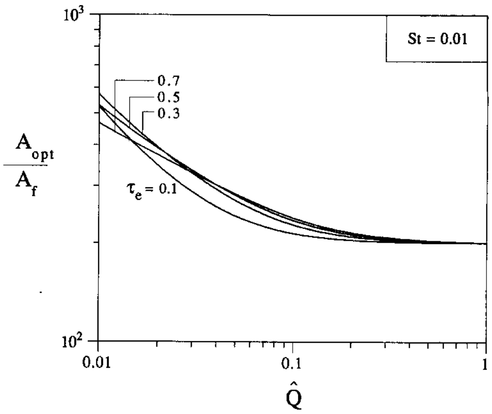

Fig. 9. The optimal heat transfer contact area for low temperature forced convection cooling with pressure drop (Section 6).

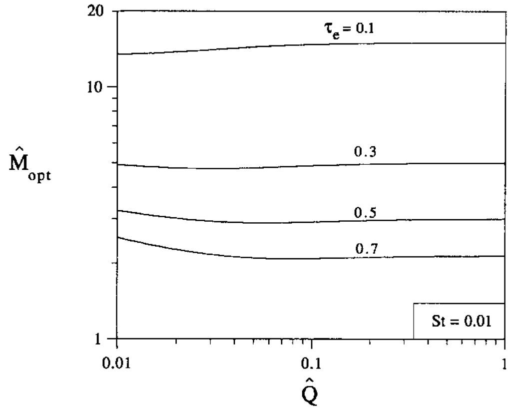

Fig. 10. The optimal coolant flow rate for low temperature forced convection cooling with pressure drop (Section 6).

The optimal refrigeration temperature $\tau_{\rm in,opt}(St, \tau_e, \hat{Q})$ associated with this second minimum can be combined with equation (40) to calculate the optimal heat transfer area $x_{\rm opt}(St, \tau_e, \hat{Q})$, as shown in Fig. 9. Finally, $\tau_{\rm in,opt}$ and $x_{\rm opt}$ can be substituted into equation (35) to determine the optimal coolant flow rate $\hat{M}_{\rm opt}(St, \tau_e, \hat{Q})$, as shown in Fig. 10. The resulting (twice-minimized) power requirement is reported in Fig. 11 for the order-of-magnitude assumption that $St = 0.01$.

One interesting feature of this double minimization of $\dot{W}$ is revealed by the analytical form of equation (40). When $\hat{Q}$ is sufficiently large, the group $St\,H/G$ on the right side of equation (40) approaches zero. In this limit $x_{\rm opt}$ approaches $2/St$, which is identical to the optimal heat transfer area $A_{\rm opt}/A_{\rm f}$ found for cooling techniques above room temperature in Sections 2–4 [see equation (12)]. This feature is confirmed by the numerical results plotted in Fig. 9, which show that $A_{\rm opt}/A_{\rm f}$ approaches $2/St$ when $\hat{Q}$ exceeds approximately 0.1.

7. CONCLUSIONS

In this paper we examined the most basic tradeoffs that govern the thermodynamic optimization of cool-

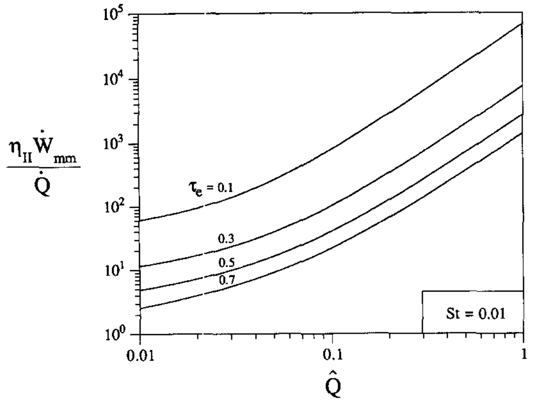

Fig. 11. The minimum refrigerator power requirement for forced convection with pressure drop (Section 6).

ing techniques for heat generating electric devices. We considered five simple models of cooling techniques for applications above and below room temperature. The key conclusions reached in this study are:

(1) In cooling techniques that rely on single-phase forced convection above room temperature, the fan power is minimum when the heat transfer contact area is optimized, as shown in equation (12).

(2) The optimal heat transfer area for minimum fan power requirement is independent of whether the air is mixed or unmixed inside the enclosure that houses the heat generating components (Section 3).

(3) In techniques with air cooling by natural draft, the heat generation rate (or amount of electronics) is maximum when the heat transfer contact area is optimized in accordance with equation (12).

(4) In forced convection cooling at low temperatures the refrigerator power requirement can be minimized in two ways, with respect to the coolant flow rate and the heat transfer contact area.

(5) The optimal heat transfer area for minimum refrigerator power requirement agrees in an order of magnitude sense with the estimate provided by equation (12).

If these optimization principles are present in models as simple as those of Figs. 1–4, then we can be sure that the same tradeoffs can be found in the more complex models used in the design of real cooling systems. Indeed, as shown in this paper and ref. [15], the mission of the simple models is to show the way, i.e. to uncover new optimization opportunities for the more applied work that will follow.

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